2024 CMWMC Relay Round Solutions

1.1. The sum of the digits of positive integer N is 55. Find the least possible sum of the digits of $N + 111$.

Proposed by Justin Hsieh

Answer. 4

Solution. Every positive integer is congruent to the sum of is digits mod 9. We have $N \equiv 55 \equiv 1 \pmod 9$. Then $N + 111 \equiv N + 3 \equiv 4 \pmod 9$. The least possible sum of digits of $N + 111$ is thus $\boxed{4}$, achieved by $1,999,999 + 111 = 2,000,110$.

1.2. Let T denote the answer to the previous problem. There exists a unique ordered pair (a, b) of positive integers with $b < T$ and

$$a + \frac{b}{T} = \frac{3}{2} \times \frac{ab}{T}.$$

What is ab?

Proposed by Justin Hsieh

Answer. 18

Solution. Multiply the equation by $2T$ to get $2Ta + 2b = 3ab$. We can solve for a: $a = \dfrac{2b}{3b - 2T}$. Once we receive T, we can try values of b until we get an integer. In our case, $T = 4$, and $b = 3$ is the only choice that makes the denominator positive. We get $a = 6$, so $ab = \boxed{18}$.

1.3. Define the *middle common factor* of three positive integers a, b, c to be

$$\text{mcf}(a, b, c) = \frac{abc}{\gcd(a, b, c) \cdot \text{lcm}(a, b, c)}.$$

For example, $\text{mcf}(10, 12, 40) = \dfrac{4800}{2 \cdot 120} = 20$.

Let T denote the answer to the previous problem. As n ranges over the positive integers, find the number of possible values for $\text{mcf}(40, T, n)$.

Proposed by Justin Hsieh

Answer. 18

Solution. The middle common factor can be computed by taking the median exponent on each prime factor in a, b, c. For each prime p, suppose that a, b, c have x, y, z factors of p, respectively. Then $\gcd(a, b, c)$ has $\min(x, y, z)$ factors of p, and $\text{lcm}(a, b, c)$ has $\max(x, y, z)$ factors of p, so $\text{mcf}(a, b, c)$ has

$$(x + y + z) - \min(x, y, z) - \max(x, y, z) = \text{median of } x, y, z$$

factors of p. In the given example, we have $10 = 2 \cdot 5$, $12 = 2^2 \cdot 3$, and $40 = 2^3 \cdot 5$. Then

$$\begin{aligned}
\gcd(10, 12, 40) &= 2, \\
\operatorname{mcf}(10, 12, 40) &= 2^2 \cdot 5 = 20, \\
\operatorname{lcm}(10, 12, 40) &= 2^3 \cdot 3 \cdot 5 = 120.
\end{aligned}$$

Turning to $\operatorname{mcf}(40, T, n)$, the factorization of 40 is $2^3 \cdot 5$, and the factorization of $T = 18$ is $2 \cdot 3^2$. As n ranges over the positive integers, the median exponent on 2 can be anything from 1 to 3, the median exponent on 3 can be anything from 0 to 2, and the median exponent on 5 can be 0 or 1. The median exponent on all other primes is 0. This gives $3 \cdot 3 \cdot 2 = \boxed{18}$ possible values for the mcf.

2.1. *Proposed by Robert Trosten*

Answer. -1

Solution. Observe by symmetry that for every nonnegative integer ℓ we have

$$\sum_{k=8\ell+1}^{8(\ell+1)} \sin\left(\frac{k\pi}{4}\right) = 0,$$

either by viewing $\sin(\theta)$ as the imaginary part of $e^{i\theta}$ or just by direct computation (and 2π-periodicity). Now either leverage 2π-periodicity of sine starting from $\frac{3\pi}{2}$ or use rotational symmetry of the $e^{i\theta}$ angles to see that

$$\sum_{k=0}^{2024} \sin\left(\frac{3\pi}{2} + \frac{\pi k}{4}\right) = \sin(3\pi/2) + \sum_{k=1}^{8(253)} \sin\left(\frac{3\pi}{2} + \frac{\pi k}{4}\right) = 0$$

or use the sine addition formula $\sin(\theta + \psi) = \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\psi)$ to find

$$\sum_{k=0}^{2024} \sin\left(\frac{3\pi}{2} + \frac{\pi k}{4}\right) = -\sum_{k=0}^{2024} \cos(\pi k/4) = -1 + \sum_{k=1}^{8(253)} \cos(\pi k/4) = \boxed{-1}.$$

Justifying that last manipulation is done by the exact same symmetry argument / direct computation as the one for $\sum_{k=8\ell+1}^{8(\ell+1)} \sin(k\pi/4) = 0$.

2.2. *Proposed by Robert Trosten*

Answer. 5

Solution. Observe that the sum of the coefficients of the polynomial $x^3 - Tx^2 + T - 1$ is 0, so 1 is a root. Dividing by $x - 1$ yields a factorization

$$x^3 - Tx^2 + T - 1 = (x-1)(x^2 + (1-T)x + (1-T)) = (x-1)(x^2 + 2x + 2)$$

upon receipt of $T=-1$. The complex roots of the quadratic are $-1+i$ and $-1-i$, each having magnitude $\sqrt{2}$, so our final answer is $1+(\sqrt{2})^2+(\sqrt{2})^2=\boxed{5}$.

2.3. *Proposed by Robert Trosten*

Answer. 7080

Solution. The initial tax on the car is $63720\cdot r$, so Irene would have to give this much money. The tax on this would be $63720\cdot r^2$, so Irene would need to give this much as well. Inductively, Irene would need to give a monetary value of

$$63720(r+r^2+r^3+\cdots)=\frac{63720r}{1-r}.$$

Upon receipt of $T=5$, we see $r=\frac{1}{10}$ and hence $\frac{r}{1-r}=\frac{1}{9}$, so $s=\boxed{7080}$.

For an alternate solution, you can reframe the question as "after taxes, Irene wants Robin to have money worth $63720r$ to cover the tax on the car." The money Robin will have after taxes is $s(1-r)$, so we equate the two and get the same answer of 7080.

3.1. *Proposed by Michael Duncan*

Answer. 12

Solution. We can split into two cases, either there is 1 letter between the two C's or there are 3. If there is 1 letter, then it can either be an I or an M. If it is an I, then there are 3 possible such strings, and if it is an M, there are $2\cdot 3=6$ possible such strings, so there are 9 possible strings in this case. Otherwise, there is 3 letters in between the C's, and there are 3 possible ways to arrange MMI in between the two C's, so there are 3 possible strings in this case. Thus the answer is $9+3=12$.

3.2. *Proposed by Michael Duncan*

Answer. 6

Solution. Suppose there are T colors. There are two cases when coloring the edges of the square, either the top and bottom edge are the same color, or they are different. In this first case, there are $T\cdot(T-1)^2$ ways to color the square. In this second case, there are $T(T-1)(T-2)^2$ ways to color the square. So the total number of colorings is $T(T-1)^2+T(T-1)(T-2)^2$. Given that $T=12$, the number of colorings X is $12\cdot 11^2+12\cdot 11\cdot 10^2=12\cdot 11\cdot(11+100)=132\cdot 111$. Thus we have $\lfloor\frac{X}{2220}\rfloor=\lfloor\frac{132}{20}\rfloor=\lfloor 6.6\rfloor=6$.

3.3. *Proposed by Michael Duncan*

Answer. 521

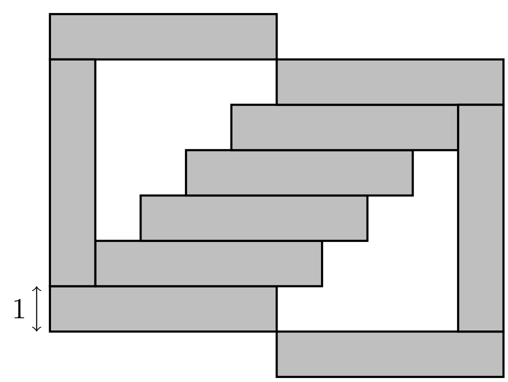
Solution. First we need to find the optimal strategy. Any strategy is of the form "stop when you catch n monsters", so we just need to find what n is. Since $T = 6$, we have $\frac{9}{16}$ probability of catching a monster, $\frac{3}{16}$ of not catching a monster, and $\frac{1}{4}$ of losing all of our monsters and stopping early. If we currently have k monsters, the expected gain after an encounter is $\frac{9}{16}(k+1) + \frac{3}{16}(k)$. We only want to do another encounter if this value is greater than k. Algebra shows that this is only true if $k < \frac{9}{4} = 2.25$, i.e. once we catch 2 monsters, it does not make sense to perform more encounters, so this is our value of n.

Next, we simply compute the expected number of turns it takes to execute this strategy. Let X_0 denote the number of steps it takes when we currently have 0 monsters, and X_1 to be the number of steps it takes when we currently have one monster. We want to find $E[X_0]$. Note that

$$E[X_0] = \frac{9}{16}(1 + E[X_1]) + \frac{3}{16}(1 + E[X_0]) + \frac{1}{4}(1)$$
$$E[X_1] = \frac{9}{16}(1) + \frac{3}{16}(1 + E[X_1]) + \frac{1}{4}(1)$$

Solving this system yields that $E[X_0] = \frac{352}{169}$ so our final answer is $352 + 169 = 521$.

4.1. All rectangles in the below figure are congruent. What is the area of the shaded region?



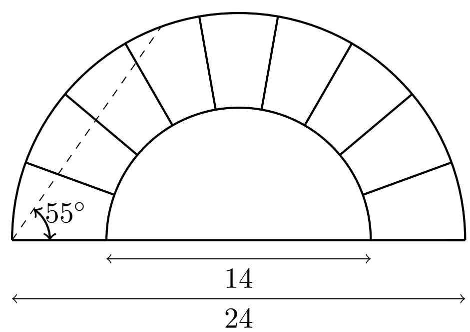
Proposed by Lohith Tummala

Answer. 50

Solution. The width of each rectangle is 1. We need to find the length. Notice that the length of each of the vertical rectangles is equal to the widths of five rectangles stacked on each other. Thus, the width is 5. There are ten rectangles, so the area is $10(1 \cdot 5) = 50$.

4.2. Let T denote the answer to the previous problem. Hamerschlag Hall is renovating its windows to a new design. The window shape is a semicircle of diameter 24 inches, but is constructed by gluing many panes. Specifically, it uses $T+1$ panes: one semicircular pane of diameter 14 inches, and T congruent panes that fill in the area outside the semicircular pane.

While no one was looking, a ninja uses her sword to make a straight line cut through a renovated window such that it makes a $55°$ angle with the horizontal and cuts through the window corner, thus damaging a bunch of panes. The total area of panes that need to be replaced is $k\pi$. Find k. (Below is an example window with $T = 9$. In this case, four panes need to be replaced since the sword cut goes through four panes).



Proposed by Lohith Tummala

Answer. 19

Solution. Let A, B be the points at which the sword cut enters and leaves the window, and let O be the center of the circle. Draw $\overline{BO}$. Since $\overline{AO} \cong \overline{BO}$, $\angle ABO = \angle BAO = 55°$, and therefore, $\angle ABO = 70°$. The T "side panes" span $180°$ of the circle. Thus, the number of side panes that span only $70°$ is $\lceil \frac{70T}{180} \rceil$. (We have to round up since a cut could partially hit the last pane as it leaves B, but we still have to replace an integer number of side panes.)

The area of all side panes is $\frac{\pi}{2}(144 - 49) = \frac{95\pi}{2}$. The area of one side pane is therefore $\frac{95\pi}{2T}$. Thus, the area of $\lceil \frac{70T}{180} \rceil$ side panes is

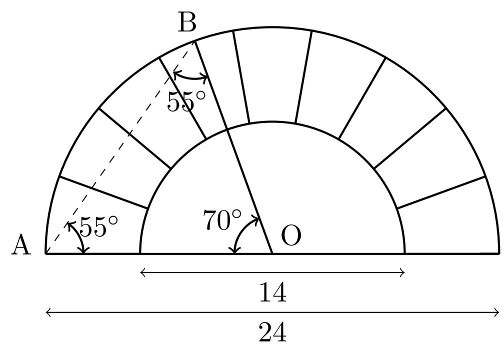
$$\lceil \frac{70T}{180} \rceil \cdot \frac{95\pi}{2T}$$

Once we know that $T = 50$, we just plug in:

$$\lceil \frac{70(50)}{180} \rceil \cdot \frac{95\pi}{2(50)}$$

$$20 \cdot \frac{19\pi}{20} = 19\pi$$

Thus, our answer is 19.



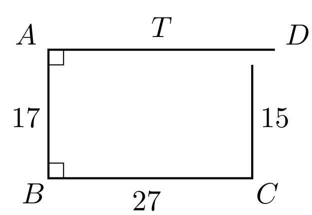
4.3. Let T denote the answer to the previous problem. A quadrilateral $ABCD$ has $AB = 17$, $BC = 27$, $CD = 15$, and $DA = T$. $ABCD$ has exactly two right angles. What is the area of $ABCD$?

Proposed by Lohith Tummala

Answer. 345

Solution. Without T, we could try all possible locations of right angles, and then find a few possible values for T and the area.

(a) $\angle A = \angle B = 90°$: This implies that $\overline{AD}$ and $\overline{BC}$ are parallel, with a distance of 17 between them. Having $\overline{CD} = 15$ would therefore be impossible, making this case impossible.

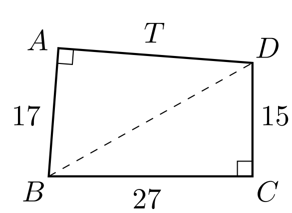


(b) $\angle A = \angle C = 90°$: This means that $\triangle BAD$ and $\triangle BCD$ are right, each sharing hypotenuse $\overline{BD}$. Thus, we have

$$BD = \sqrt{17^2 + T^2} = \sqrt{27^2 + 15^2} \implies T = \sqrt{665}$$

Thus, the area of $ABCD$ is

$$\frac{1}{2}(17\sqrt{665} + 27 \cdot 15) = \frac{17\sqrt{665} + 405}{2}$$

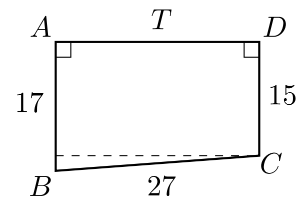


(c) $\angle A = \angle D = 90°$: This implies that $\overline{AB}$ and $\overline{CD}$ are parallel, so we have a trapezoid. From the figure below, we can find

$$T = \sqrt{27^2 - 2^2} = 5\sqrt{29}$$

The area is therefore

$$16 \cdot 5\sqrt{29} = 80\sqrt{29}$$

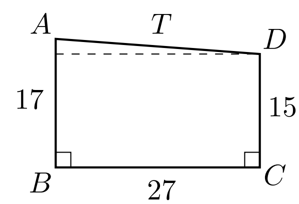


(d) $\angle B = \angle C = 90°$: This implies that $\overline{AB}$ and $\overline{CD}$ are parallel, so we have a trapezoid. The value is T can be computed via the following:

$$T = \sqrt{27^2 + 2^2} = \sqrt{733}$$

The area is

$$16 \cdot 27 = 432$$

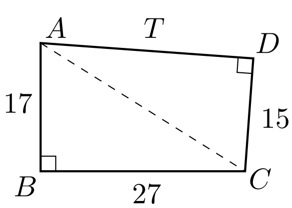


(e) $\angle B = \angle D = 90°$: This means that $\triangle CAD$ and $\triangle CAB$ are right, each sharing hypotenuse $\overline{AC}$. Thus, we have

$$AC = \sqrt{17^2 + 27^2} = \sqrt{T^2 + 15^2} \implies T = \sqrt{793}$$

Thus, the area of $ABCD$ is

$$\frac{1}{2}(15\sqrt{793} + 27 \cdot 17) = \frac{15\sqrt{793} + 459}{2}$$

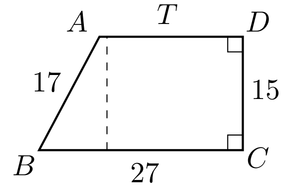


(f) $\angle C = \angle D = 90°$: This implies that $\overline{AD}$ and $\overline{BC}$ are parallel, so we have a trapezoid. From the figure below, we can find

$$T = 27 - \sqrt{17^2 - 15^2} = 27 - 8 = 19$$

The area is therefore

$$15 \cdot \frac{19+27}{2} = 345$$



We have five possible values of T. Once we know that $T = 19$, we see that only the last case is valid, giving us an area of 345.